Section 1.1

Introduction:

We will mainly use two problems to illustrate some of the basic ideals of differential equations.

The main reason differential equations show up in the description of the real world is that many of the principles or laws underlying the behavior of the natural world are relations involving "rates". In mathematical terms,

relations $\rightsquigarrow$ equations

rates $\rightsquigarrow$ derivatives.

Equations containing derivatives are differential equations.

Example 1: Suppose that an object is falling in the atmosphere near the sea level. Formulate a differential equation that describes the motion.

We let v denote the velocity of the object, let t represen[t] the time. And then v is a function of t.

The physical law that governs the motion of objects in Newton's second law: The net force on the object = Mass · accelerati[on]

$$F = ma$$

Now a is ~~not~~ just the derivative of v w.r.t the time t.

Next we consider the forces that act on the object:

(1). Gravity: gravity exerts a force equal to mg, where g is the acceleration ~~due~~: $g = 9.8\,m/sec^2$.

(2). Air resistance: A simplified description of air resistance is that it's proportional to the velocity, given by $\gamma \cdot v$, where γ is a constant called the drag coefficient.

The net force acting on the falling object is

$$F = mg - \gamma v,$$

where the negative sign comes from the fact that it's upward.

Remark: 1. The drag coefficient depends on the material, and shape of the object.

2. Deeper thinking: Sometimes, the mass can also be a ~~time~~ function of time t.

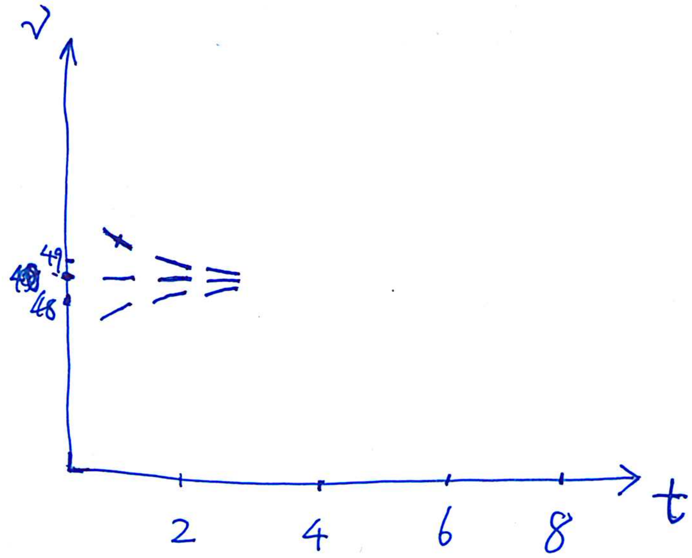
~~(3). The~~

Let us give a specific version of the above equation:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

Let us describe a geometrical ~~point of~~ viewpoint of the above equation.

Let us draw a the v-t plane:



According to the equation:

If $v = 50 \Rightarrow \frac{dv}{dt} = -0.2$

Recall that $\frac{dv}{dt}$ is the slope. We can draw line segments for other v's. This then is an example of a direction field.

Suppose $v(t)$ is a solution of the differential equation, then at each ~~$(v(t)$~~ $(t, v(t))$; the line segment is the tangent line of the graph of the function $V(t)$. This is a geometric ~~ex~~ explanation of a solution.

Equilibrium solution:

The constant function $V(t) = 49$ is a solution of the equation. Since it does not change w/ the time t, it's called an <u>equilibrium solution.</u>

Observation: As $t \to \infty$, all other solutions seem to be converging to the equilibrium solution.

Direction fields: For the studing of solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \text{ where } f \text{ is}$$

a given function of t and y, a direction field can be drawn on the t-y plane.

Such a direction field gives a good picture of the overall behavior of solutions of a differential equations.

Remark: You can imagine how a computer can draw such a direction field, and approximate some solutions.

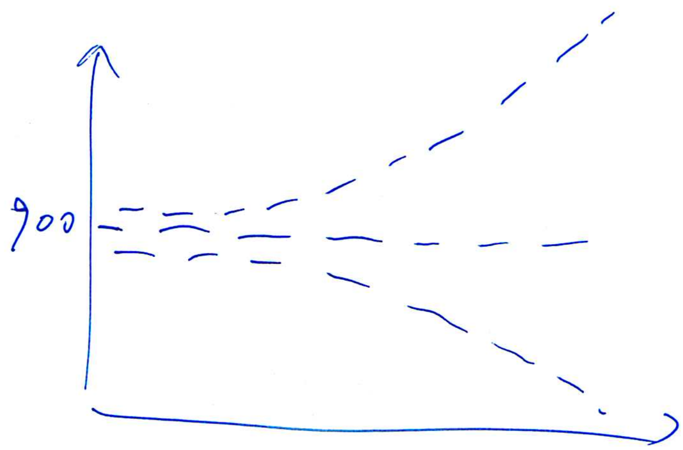
Example 2: Consider a population of field mice in a certain area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. (Say 0.5 mice/month) (5).

Now suppose there are several owls live in the same area and that they kill 15 field mice per day. (450 per month).

Then the equation describing the population is

$$\frac{dp}{dt} = 0.5p - 450.$$

The direction field is not the same as falling object:



Remark: ~~In the real wor~~ This model is over-simplified.

(Explain why)

~~Ex~~ To generalize, there are the following steps in constructing mathematical models:

1. Identify the dependent and independant variables. Usually the independent variable is time t.
2. Choose the ~~correct~~ units of measurements.
3. Articulate the basic law or principle that governs the problem. (This is not mathematical).
4. Express the principle or law in step 3 in terms of the variables in step 1.

Section 1.2: Solutions of some differential equations:

Example 1: Consider the equation $\frac{dp}{dt} = 0.5p - 450$, which describes the population of field mice.

$\Rightarrow$ For this, we write in the form

$$\frac{dp}{dt} = \frac{p-900}{2}.$$

$$\Rightarrow \frac{dp}{dt} \Big/ p-900 = \frac{1}{2}.$$

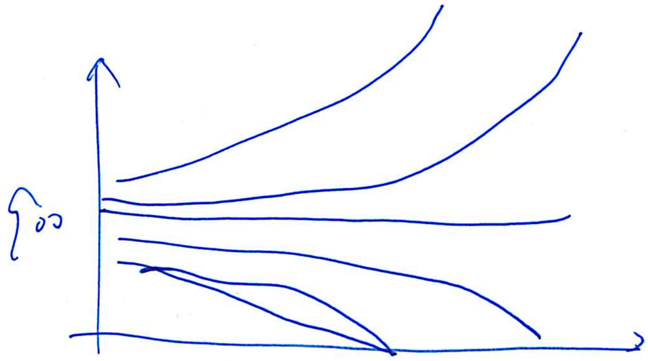
If $p \neq 900$, ~~then~~ Then. $\frac{d}{dt} \ln|p-900| = \frac{1}{2}$

$$\Rightarrow \ln|p-900| = \frac{t}{2} + C.$$

$$\Rightarrow |p-900| = e^{\frac{t}{2}} \cdot e^{C}.$$

and $p = 900 + ce^{t/2}$.

Note that there are infinite number of solutions of the differential equation.



Usually, we want to focus on a single member of the infinite family of solutions by specifying the value of the constant.

~~For~~ Most often, we ~~and often~~ do this by specifying ~~the value of the arbitrary~~ a point that must lie on the graph of the solution.

In the above equation, we could require the population have a given value at a certain time, such as value 850 at $t=0$.

Then the solution is

$$p = 900 - 50e^{t/2}.$$

The condition $p(0)=850$ we used to determine c is an example of an <u>initial condition</u>. The differential equation w/ the initial condition form an initial value problem.

Remark:

In this way, we can say that the solution of the initial value problem predicts the population for time t.

We can consider the more general problem of the differential equation

$$\frac{dy}{dt} = ay - b \quad \text{and initial condition } y(0) = y_0,$$

where y_0 is an arbitrary initial value.

Suppose $a \neq 0$, $y \neq b-a$.

Then the above equation can be written as

$$\frac{dy/dt}{\cancel{a}y - \frac{b}{a}} = a.$$

$$\Longrightarrow \quad \ln\left|y - \frac{b}{a}\right| = a \cdot t + C.$$

$$y - \frac{b}{a} = c \cdot e^{at}$$

$$\Longrightarrow \quad y = \frac{b}{a} + c \cdot e^{at}$$

And the initial condition gives rise to

$$y_0 = \frac{b}{a} + C$$

$$\Longrightarrow \quad C = y_0 - \frac{b}{a}$$

and $$y = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) \cdot e^{at}$$

Remark:
The geometrical representation of the general solution is an infinite family of curves called <u>integral curves</u>.

Section 1.3: Classification of Differential Equations:

Motivation: We want to classify D.E's and study them from the simple ones.

ODE's and PDE's:

If only ordinary derivatives appear in the differential equation, it is said to be an ordinary differential equation (ODE).

If the derivatives are partial derivatives then the equation is called a PDE.

Example of PDE: The heat conduction equation:

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}.$$

Here $u(x,t)$ denotes the temperature at a point x on an object ~~at~~ t.

Systems of Differential Equations:

If there is a single function to be determined, then one equation is sufficient. If there are ~~two~~ or more unknown functions, a system of equations is required.

Example: Lotka–Volterra or predator–prey model:

$$\frac{dx}{dt} = ax - \alpha xy$$

$$\frac{dy}{dt} = -cy + \gamma xy,$$

where $x(t)$ and $y(t)$ are the respective populations of the prey and predator species.

(Explain the sign $a, -c$.)

Remark: ODE, PDE ~~depen~~ differs by the number of independents.

Equation or Equations differ by the number of dependents.

Order of differential equations:

The order of a differential equation is the order of the highest derivative that appears in the equation.

Example: The heat equation $\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$ is a second-order PDE.

In general, the equation

$$F(t, u(t), u'(t), \cdots, u^{(n)}(t)) = 0$$

is an ODE of order n.

Example: $y''' + 2e^{t^2} y'' + (y')^2 = t^4$ is a third order ODE.

Linear and non-linear equations:

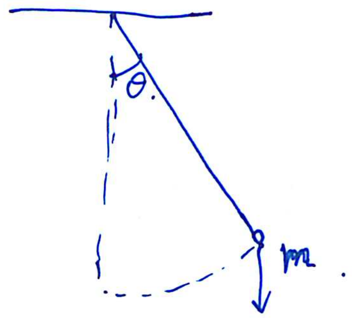
The ~~ordinary~~ ODE

$$F(t, y, y', \cdots, y^{(n)}) = 0$$ is said to be <u>linear</u>

if F is a linear function of the variables $y, y', \cdots, y^{(n)}$. Thus the general linear ~~eq~~ ODE of order n is

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_n(t) \cdot y = g(t).$$

Example of non-linear ODE:



The equation describing the motion of a pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \cdot \sin\theta = 0.$$

The presence of $\sin\theta$ makes the equation non-linear.

The linearization is given by

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \quad (\text{for } \theta \text{ small enough}).$$